Analytical Approach to Orbit Determination in the Presence of Model Errors

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The results of a study are presented in which a spacecraft's acceleration is separated into two components so that one component is tractable to analytical solution and the remaining component is treated as an unmodeled acceleration. By this procedure, an analytical solution to the dynamical equations used in the extended Kalman filter can be obtained. Two cases are considered: 1) the modeled portion is taken to be the problem of two bodies with all perturbations treated as a model error, and 2) the modeled portion is taken to be a straight line with all accelerations treated as a model error. The model error is approximated by a first-order Gauss-Markov process. The approach is evaluated using computer simulations of a lunar orbiter, where the primary model error sources are lunar surface mascons. The simulation shows that the estimates of the unmodeled mascon acceleration as obtained by the two-body model and the Gauss-Markov process agree very well with the full numerical integration but the straight line model and the Gauss-Markov process performs less accurately.

Introduction

THE effect of model errors on orbit determination procedures is well known. If sufficiently large model errors exist in the dynamical equations, it is not possible to obtain estimates of the state commensurate with the observation accuracy using either the classical least squares (or batch processor) or the sequential processor. In the presence of model errors, the differences between the observed quantity and the computed value often exhibit periodic or pseudoperiodic characteristics.

The Lunar Orbiter series and the Apollo program provided some impetus in studying ways of estimating the accelerations not modeled in the dynamical equations and simultaneously improving the estimate of the satellite's state. Muller and Sjogren[1,2] fitted the Lunar Orbiter range-rate residuals with a smooth curve which was then differentiated to obtain the unmodeled acceleration along the line of sight which resulted in a mascon representation of the lunar gravity anomalies. A somewhat different approach was taken by Tapley and Ingram[3,4] using a sequential estimation procedure which compensated for the model errors. In this approach, referred to in this paper as the Dynamic Model Compensation (DMC) method, the model errors are assumed to consist of a time correlated component and a purely random component. The model errors are then approximated by a first-order, stationary, Gauss-Markov process. Application of the DMC method to the lunar orbit phase of Apollo 10 and 11 by Ingram and Tapley[4] showed that the DMC method yielded observation residuals within the observation noise level and provided estimates of the acceleration due to the model error which could be correlated with the mascons reported by Muller and Sjogren.[1]

Additional applications of the DMC method were made by Schutz et al.[5] to a near-Earth satellite problem. Recently, Tapley and Schutz[6] considered the lunar satellite using computer simulation and showed that the DMC method can provide estimates of the unmodeled acceleration which are representative of the true acceleration. In addition, it was shown that the accuracy of the acceleration estimate depends on the magnitude of the unmodeled acceleration and the accuracy of the observations. The dynamical equations for the estimation algorithm were solved using numerical integration.

Recognizing that the representation of the unmodeled acceleration by a first-order Gauss-Markov process does not require the acceleration to arise from certain sources, the representation can be used to handle accelerations due to venting, drag, and other nonconservative forces as well as conservative forces. Furthermore, forces which are well known could be treated as an unmodeled acceleration, i.e., not modeled in the dynamical equations.

By separating a spacecraft's acceleration into two components so that one component is tractable to analytical solution and the remaining component is treated as an unmodeled acceleration, an analytic solution to the dynamical equations can be obtained. Two specific cases are considered in this paper. In the first case, the modeled motion arises from the two-body central force field and all perturbations are treated as unmodeled accelerations along with the "actual" unmodeled acceleration. Evaluation of this method is performed using a lunar satellite in which the "actual" unmodeled accelerations are due to four point masses located on the lunar surface. The perturbations, normally modeled, are J_2 and the Earth. In the second case, the two-body force is treated as an unmodeled acceleration; thus, the modeled motion is unaccelerated, straight-line motion.

Jazwinski[7,8] has taken an alternate approach using the J-adaptive filter. This estimator allows rapid orbit determination and geopotential model improvement and differs from the DMC method partially through the representation of the model errors. The method discussed in the subsequent sections is a variation of the DMC method which requires no numerical integration and offers a fast solution to the dynamical equations.

Estimation of Unmodeled Accelerations

The differential equations of motion for a satellite can be expressed as

$$\dot{r}_i = v_i, \quad \dot{v}_i = a_i + m_i, \quad i = 1, 2, 3 \tag{1}$$

where the subscript i represents the axes of a Cartesian coordinate system, a_i represents a modeled acceleration component along axis i, and m_i is the effect of all accelerations not included in the

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mathematical model. Subsequently, m_i will be referred to as the "unmodeled acceleration."

In the DMC estimation procedure, the unmodeled accelerations, m_i, are approximated as a stationary, first-order, Gauss-Markov process, $\xi_i(t)$, which satisfies

$$\dot{\xi}_i(t) = -\xi_i(t)/T_i + u_i(t) \tag{2}$$

where $u_i(t)$ represents Gaussian noise with statistics $E[u_i(t)] = 0$ and $E[u_i(t)u_j(\tau)] = q_{ij}(t)\,\delta(t-\tau)$ where $\delta(t-\tau)$ is the Dirac delta function and $t \neq \tau$. In the DMC method, the time correlation coefficients, T_i, are assumed to be unknown parameters whose values are to be determined during the estimation process. If ξ_i is substituted for m_i in Eq. (1), then the differential equations for the state of the dynamical system become

$$\dot{X} = F(X, u, t) \tag{3}$$

where the state vector X contains the satellite position and velocity components (r_i, v_i), the components of the unmodeled acceleration (ξ_i), and the time correlation coefficients (T_i). Equation (3) is used to obtain the reference orbit for which the initial conditions, X_o, are only approximately known.

The relationship between the p-dimensional observation vector, Y_j, the observation noise, v_j, and the state, X_j, at time t_j is $Y_j = G(X_j, t_j) + v_j$. The observation noise statistics are assumed to be $E[v_j] = 0$ and $E[v_j v_k^T] = D_j \delta_{jk}$ where δ_{jk} is the Kronecker delta.

The problem considered is to determine the best estimate of the state in the minimum variance sense given the relation for propagating the state, Eq. (3), the observation-state relation, and the sequence of observations, Y_j. Under these conditions, the estimate of the state, $\hat{X}_j$, can be obtained using the extended form of the Kalman estimator.[9] The state transition matrix, $\Phi(t_j, t_{j-1})$, is required by the estimator and satisfies the differential equation $\dot{\Phi}(t, t_j) = A(t)\Phi(t, t_j)$ where $A(t) = [\partial F/\partial X]^*$. The symbol $[\ ]^*$ indicates that the quantity in the brackets is evaluated on the reference solution, i.e., the solution to Eq. (3) usually obtained by numerical integration. Since the extended Kalman filter is used, the reference orbit is updated or rectified at each observation time to the current estimate of the state, $\hat{X}_j$. This estimate includes an estimate for the state of the satellite as well as the unmodeled accelerations and the time correlation coefficients. A priori estimates of these quantities are required in addition to the state error, the observation noise, and the state noise covariance matrices. The latter matrix is given in analytical form by Ingram.[10]

Analytical Solutions

Two-Body Model

The method discussed in the previous section for estimating unmodeled acceleration was developed to handle model errors such as those arising due to inadequate knowledge of a planet's gravitational field. Since no restrictions are placed on the source of the model error which is approximated by a Gauss-Markov process, even accelerations which are well known could be treated as a model error.

The equations of motion for a satellite can be written as

$$\ddot{r}_i = -\mu r_i/r^3 + f_i + \xi_i' \tag{4}$$

where r_i is the satellite's position component along axis i on a reference orbit relative to a central attracting body with a gravitational parameter μ. In Eq. (4), f_i is a component of the vector acceleration containing all modeled effects and ξ_i' represents all unmodeled effects. The osculating orbital elements at a time t_o define a two-body orbit which yields a position component R_i at a later time, t_j. This position on the two-body reference is related to the position by $r_i = R_i + \eta_i$. Substitution into Eq. (4) yields

$$\ddot{\eta}_i = -\ddot{R}_i - \mu r_i/r^3 + f_i + \xi_i' \tag{5}$$

Letting

$$\Delta\xi_i' = -\mu r_i/r^3 - \ddot{R}_i \tag{6}$$

then the differential equation for the deviation from the two-body reference orbit is

$$\ddot{\eta}_i = f_i + \Delta\xi_i' + \xi_i' \tag{7}$$

If both f_i and $\Delta\xi_i'$ are treated as an unmodeled acceleration, then

$$\ddot{\eta}_i = \xi_i \tag{8}$$

where $\xi_i = f_i + \Delta\xi_i' + \xi_i'$. Thus ξ represents all modeled perturbations as well as unmodeled accelerations and the error due to approximating $\ddot{R}_i$ by $-\mu r_i/r^3$. Integration of Eq. (8) yields η_i and $\dot{\eta}_i$ which, when added to the position and velocity, respectively, on the two-body orbit, yields the reference orbit for the estimation algorithm.

As discussed in the previous section, the unmodeled acceleration is represented by a stationary, first-order, Gauss-Markov process, viz., $\dot{\xi}_i = -\xi_i/T_i$, where a nonrotating coordinate system centered at the central attracting body is used. Integration of ξ yields $\xi_i = \xi_{i_o}\alpha_i$ where $\alpha_i = \exp(-\Delta t/T_i)$, $\Delta t = t - t_o$, and $\xi_{i_o} = \xi_i(t_o)$. Thus, Eq. (8) can be integrated to yield the deviation from the two-body orbit

$$\begin{aligned} \dot{\eta}_i &= -T_i\xi_{i_o}\beta_i + \dot{\eta}_{i_o} \\ \eta_i &= \eta_{i_o} + T_i\xi_{i_o}(T_i\beta_i + \Delta t) + \dot{\eta}_{i_o}\Delta t \end{aligned} \tag{9}$$

where $\beta_i = \alpha_i - 1$. In the extended form of the Kalman filter, the reference is updated or rectified to the current estimate of the state. This estimate defines a new two-body orbit and is analogous to the rectification performed in Encke's method. By rectifying, $\eta_{i_o} = 0$ and $\dot{\eta}_{i_o} = 0$ each time an observation or group of simultaneous observations is processed. Since both the position on the two-body reference, R_i, and the deviation from it, η_i, can be obtained analytically, the position, r_i, and velocity, v_i, on the reference orbit can be obtained without numerical integration.

The estimation procedure also requires the 12×12 state transition matrix, Φ, which can be expressed as $\partial X/\partial X_o$. Since $r_i = R_i + \eta_i$, it follows that, for example, $\partial r_i/\partial r_{i_o} = \partial R_i/\partial r_{i_o}$, and the 6×6 submatrix in the upper left-hand corner of Φ is simply the two-body state transition matrix. The 12 remaining 3×3 submatrices of Φ are all diagonal matrices for which the diagonal elements are

$$\begin{aligned} \partial r_i/\partial \xi_{i_o} &= T_i(T_i\beta_i + \Delta t) \\ \partial r_i/\partial T_i &= \xi_{i_o}[2T_i\beta_i + \Delta t(\alpha_i + 1)] \\ \partial \dot{r}_i/\partial \xi_{i_o} &= -T_i\beta_i \\ \partial \dot{r}_i/\partial T_i &= -\xi_{i_o}[\beta_i + \Delta t\,\alpha_i/T_i] \\ \partial \xi_i/\partial r_{i_o} &= \partial \xi_i/\partial \dot{r}_{i_o} = \partial T_i/\partial r_{i_o} = \partial T_i/\partial \dot{r}_{i_o} = \partial T_i/\partial \xi_{i_o} = 0 \\ \partial \xi_i/\partial \xi_{i_o} &= \alpha_i \\ \partial \xi_i/\partial T_i &= \xi_{i_o}\Delta t\alpha_i/T_i^2 \\ \partial T_i/\partial T_{i_o} &= 1 \end{aligned} \tag{10}$$

Since the two-body state transition matrix can be determined analytically (for example, see Goodyear[11] and Broucke[12]), no numerical integration is required to obtain the full state transition matrix. As shown in the preceding paragraphs, the reference orbit can be expressed analytically also.

Straight-Line Model

If the entire right-hand side of Eq. (4) is treated as an unmodeled acceleration, including the central body term, then

$$\ddot{r}_i = \xi_i. \tag{11}$$

Thus, the modeled portion has zero acceleration, or unaccelerated motion in a straight line. Integration of this equation yields

$$\dot{r}_i = -T_i\xi_{i_o}\beta_i + \dot{r}_{i_o} \tag{12}$$

$$r_i = r_{i_o} + T_i\xi_{i_o}[T_i\beta_i + \Delta t] + \dot{r}_{i_o}\Delta t \tag{13}$$

In this case, however, $\dot{r}_{i_o}$ and r_{i_o} are not zero even though rectification takes place. Because of the similarity of this solution with the previous two-body case, the state transition matrix differs only in the upper left-hand 6×6 submatrix found in the

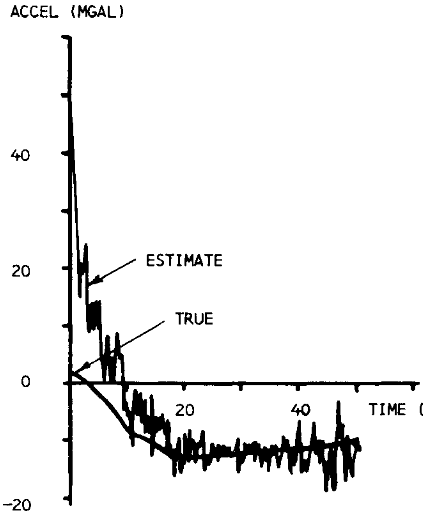

Fig. 1 ε_r-Component of mascon acceleration for 180° inclination.

previous case to be the two-body transition matrix. For the straight-line model, this 6 × 6 matrix should be replaced with

$$\Phi_{\text{straight line}} = \begin{bmatrix} 1 & 0 & 0 & \Delta t & 0 & 0 \\ 0 & 1 & 0 & 0 & \Delta t & 0 \\ 0 & 0 & 1 & 0 & 0 & \Delta t \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{14}$$

Generation of the Simulated Observations

The simulated observations of a lunar satellite are generated by numerically integrating

$$\ddot{r}_i = -(\mu/r^3)r_i + f_i + m_i \tag{15}$$

with initial conditions

$$r_i(t_o) = r_{i_o}, \quad \dot{r}_i(t_o) = \dot{r}_{i_o}$$

where f_i contains the lunar J_2 and the Earth effects and m_i is the acceleration which is unknown in the estimation procedure. In the simulation, m_i represents the acceleration due to n point-masses representing lunar surface mascons. Integration of Eq. (15) yields a solution which will be termed the "true" motion. It is used in generating the simulated observations. A range rate, $\dot{\rho}$, is generated using the true position and velocity of the satellite obtained from integrating Eq. (15). The final simulated observation is generated by adding $\sigma\lambda$ to the true $\dot{\rho}$, where σ is the standard deviation of the observation noise and λ is a Gaussian distributed random variable with zero mean and unit variance. The observation interval is assumed to be 6 sec and up to five stations can observe the satellite simultaneously. In all cases considered in this paper, σ is assumed to be 1.5 mm/sec.

To simulate the real-world, the initial conditions used in generating the observations were different from those used in the estimation. The differences between the two sets of initial conditions were: $\Delta r_1 = -306$ m, $\Delta \dot{r}_1 = 0.05$ m/sec; $\Delta r_2 = -305$ m, $\Delta \dot{r}_2 = -0.37$ m/sec; $\Delta r_3 = 72$ m, $\Delta \dot{r}_3 = 0.31$ m/sec. The satellite started at approximately 90° East selenographic longitude with an altitude of 100 km and zero eccentricity. The simulated mascon locations are shown in Table 1, taken from Ref. 2. The two orbits used are the ones used by Tapley and Schutz[6] in which both are retrograde, but one is equatorial and the other has an inclination of 150°. The latter inclination results in the satellite passing directly over two of the mascons.

The estimation of the unmodeled acceleration is performed using a selenocentric, nonrotating, Cartesian coordinate system.

Table 1 Mascon locations

Latitude (deg)	Longitude (deg)	Mascon mass ÷ Lunar mass
38	−18	20×10^{-6}
28	18	20×10^{-6}
16	58	10×10^{-6}
−16	34	9×10^{-6}

For convenience, this acceleration can also be expressed in a local spacecraft system which has unit vectors ε_r along the selenocentric position vector, ε_λ is directed due east, and ε_ϕ is along the spacecraft latitude meridian. In the estimation program, the acceleration transformation from the selenocentric system to the local spacecraft system is performed using the state estimate of the satellite.

The estimation program requires no numerical integration and uses the analytical solutions discussed in the preceding sections. The two-body state and state transition matrix solution of Goodyear[11] was used in the two-body formulation.

Results

The results discussed in this section, obtained using the analytical solutions discussed previously, will be compared with those given in Ref. 6 which were obtained using numerical integration of the dynamical equations used in the estimation algorithm. It should be noted that, if used in the classical least squares, the two-body reference orbit would yield large observation residuals. The straight-line reference would, of course, perform even less satisfactorily. Used in a sequential algorithm with no model compensation, the estimate would diverge from the true solution with either the two-body or the straight-line reference solutions. The behavior of these reference solutions in the DMC method is discussed in the following sections.

Two-Body

The simulated observations were processed using the analytical solutions of this paper for the reference orbit. In these analytical solutions, all perturbations to the central body term were treated as unmodeled accelerations. In order to compare with Ref. 6, it is necessary to extract only the effect due to the mascons. This was accomplished by subtracting from the total acceleration estimate the effect of J_2, the Earth, and the term given by Eq. (6) which were all computed using the estimate of the state at each observation time. Having removed these "modeled" accelerations, the remaining acceleration components are plotted in Figs. 1 and 2 for an inclination of 180°. Figure 1 is the ε_r-component and

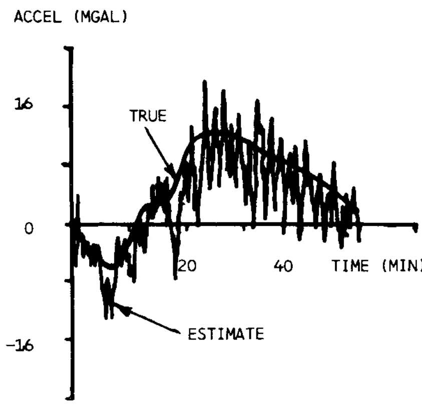

Fig. 2 ε_λ-Component of mascon acceleration for 180° inclination.

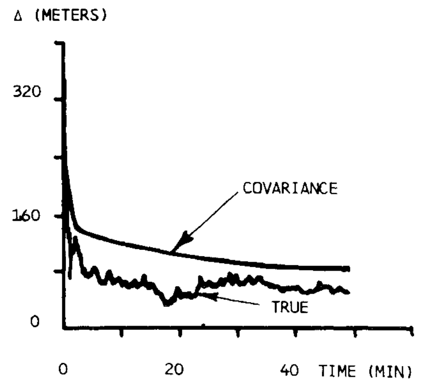

Fig. 3 Position error norm for 180° inclination.

Fig. 2 is the ε_λ-component in mgals. The ε_ϕ-component behavior is similar to that shown in Figs. 1 and 2. In these figures, the smooth line is the true simulated acceleration whereas the uneven line is the acceleration estimate. Note that the estimate follows the true very well. Figure 3 shows the true position error of the satellite (uneven line) and the square root of the trace of the position elements in the covariance matrix. It is apparent that the covariance bounds the true error, i.e., yields a conservative estimate of the error. Comparison of Figs. 1–3 with the same case in Ref. 6 shows that the estimates of the unmodeled acceleration from the analytical approach of this paper agree extremely well with the numerical integration. After the first 10 min, the initial transients in the estimate have disappeared and there is very little difference between the analytical solution and the numerical integration solution estimates of acceleration. The position error norm is comparable also.

For the 150° inclination orbit, the mascon acceleration in the simulation is considerably larger than the equatorial orbit. The results using the analytical solution are shown in Figs. 4 and 5 for the ε_r-component and the ε_λ-component, respectively. Again, the acceleration estimate follows the true acceleration extremely well and, furthermore, comparison with the numerical integration solution of Ref. 6 reveals little difference. The position error norm is shown in Fig. 6 and differs from the comparable figure of Ref. 6 by less than 10%.

Straight Line

It is shown in Ref. 6 that the accuracy with which the unmodeled acceleration can be estimated is dependent on the magnitude of the unmodeled acceleration as well as the observation accuracy. In the straight-line reference, the central body acceleration term dominates and is approximately 500 times larger than the other terms. Thus, it is not surprising that use of the straight-line solution provides good estimates of the central-body acceleration. In fact, the difference between the true acceleration and the estimate cannot be discerned when plotted. However, the acceleration estimates have a scatter on the order of 300 mgal, much too large to allow extracting the effect of the other terms, such as the mascons. The actual position error norm is also greater than predicted by the covariance matrix. Although this could be reduced by increasing the a priori value of the state noise, this would further increase the scatter making it even more difficult to extract smaller effects such as those due to the mascons.

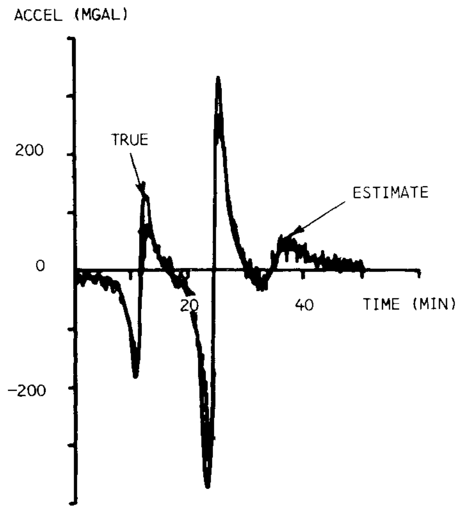

Fig. 5 ε_λ-Component of mascon acceleration for 150° inclination.

Conclusions

For the orbits and perturbations studied, estimates of the unmodeled accelerations using a two-body reference orbit can be obtained which are essentially the same as those obtained from numerical integration. The state estimate is within 10% of the numerical integration accuracy in the cases studied, but it may differ in other cases. Further study is required to ascertain bounds on the state estimate error using the analytical method.

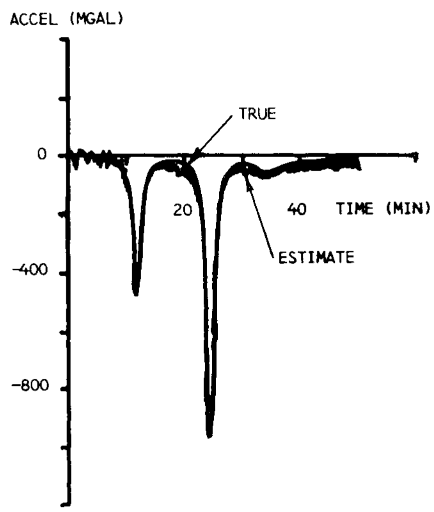

Fig. 4 ε_r-Component of mascon acceleration for 150° inclination.

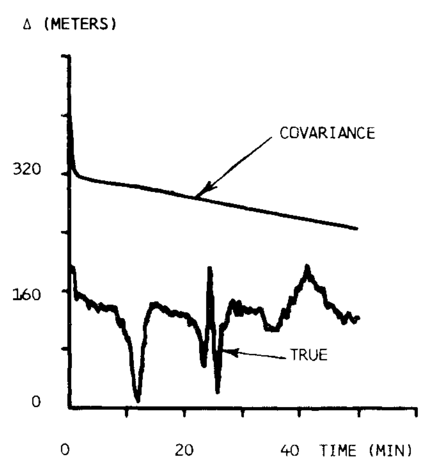

Fig. 6 Position error norm for 150° inclination.

Using the straight-line reference, the central-body acceleration can be easily estimated. However, due to large model error, smaller unmodeled accelerations cannot be detected. This, then, implies that the basic dynamics of the problem cannot be ignored in obtaining good estimates of the unmodeled acceleration with the DMC method. It should be noted that the two-body reference requires approximately one-half the computer time required for numerical integration.

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